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Note

A series of identities for the coefficients of inverse matrices on a Hamming scheme

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Abstract

In this paper, a series of identities concerned with inverse matrices of a linear combination of association matrices on Hamming schemes is given, which is useful in the field of statistical design of experiments.

1. Introduction

Association schemes were first introduced by statisticians in connection with the design of experiments [2, 5, 6].

Let $U = \{u_1, \ldots, u_v\}$ be a finite set. We assume that $n+1$ binary relations $R_0, \ldots, R_n$ are defined on the set U. Let $D_i = (d_{kl}^{(i)})$ be a $v \times v$ (0, 1)-matrix such that

$$d_{kl}^{(i)} = \begin{cases} 1 & \text{if } (u_k, u_l) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. The $(n+1)$-tuple $\langle D_0, \ldots, D_n \rangle$ is called an *association scheme* on a finite set U of v points if D_i satisfies the following conditions:

(i) D_i is symmetric for $i = 0, 1, \ldots, n$,

(ii) $\sum_{i=0}^{n} D_i = J_v$, where J_v is the $v \times v$ all-one matrix,

(iii) $D_0 = I_v$, where I_v is the $v \times v$ identity matrix,

(iv) $D_i D_j = \sum_{k=0}^{n} c_{ijk} D_k = D_j D_i$ for $i, j = 0, 1, \ldots, n$, where c_{ijk} is a constant depending on i, j and k.

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It is well-known that the number of 1's contained in a row or a column of D_i is a constant ($=v_i$) not depending on the particular choice of a row or a column. And the vector space consisting of all matrices $\sum_{i=0}^{n} a_i D_i$ is a ring (see, for example, [1, 3]). It is obvious that if an element $\sum_{i=0}^{n} a_i D_i$ has the inverse, then $(\sum_{i=0}^{n} a_i D_i)^{-1}$ can be written by a linear combination of $D_0, \ldots, D_n$, if it exists.

Let $F = \{0, 1, \ldots, q-1\}$ and $U = F^n$. For $x = (x_1, \ldots, x_n) \in F^n$ and $y = (y_1, \ldots, y_n) \in F^n$, the number of j such that $x_j \neq y_j$ is called the *Hamming distance* between x and y, denoted by $d(x, y)$. We define the relation R_i by $(x, y) \in R_i$ if $d(x, y) = i$, then $\langle D_0, \ldots, D_n \rangle$ is an association scheme, which is called a *Hamming scheme.*

In this paper, we shall obtain the explicit formula of $(\sum_{i=0}^{n} x^i D_i)^{-1}$. Furthermore, we shall give a series of identities which is concerned with the coefficients $\{b_i\}$ of the inverse matrix $\sum_{i=0}^{n} b_i D_i = (D_0 + xD_1)^{-1}$ on a Hamming scheme.

2. Identities

The vector space consisting of all matrices $\sum_{i=0}^{n} a_i D_i$ has the unique basis of primitive idempotents $E_0, \ldots, E_n$. Let $D_i E_j = p_i(j) E_j$, where $p_i(j)$, $(j = 0, \ldots, n)$ are the eigenvalues of D_i. Let

$$P = \begin{bmatrix} p_0(0) & p_1(0) & \cdots & p_n(0) \\ p_0(1) & p_1(1) & \cdots & p_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ p_0(n) & p_1(n) & \cdots & p_n(n) \end{bmatrix}$$

be the first eigenmatrix and let

$$Q = vP^{-1} = \begin{bmatrix} q_0(0) & q_1(0) & \cdots & q_n(0) \\ q_0(1) & q_1(1) & \cdots & q_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ q_0(n) & q_1(n) & \cdots & q_n(n) \end{bmatrix}$$

be the second eigenmatrix.

It is known that $p_0(j) = q_0(j) = 1$ and $p_i(0) = v_i$. In the case of Hamming scheme, $v = q^n$ and $p_i(0) = v_i = \binom{n}{i}(q-1)^i$ hold.

For an association scheme, the following proposition is obtained.

Proposition. *For an association scheme $\langle D_0, \ldots, D_n \rangle$ on U, let $(\sum_{k=0}^{n} a_k D_k)^{-1} = \sum_{i=0}^{n} b_i D_i$, then*

$$b_i = \frac{1}{v} \sum_{j=0}^{n} \frac{q_j(i)}{\sum_{k=0}^{n} a_k p_k(j)} \tag{1}$$

holds and we have

$$\sum_{i=0}^{n} b_i p_i(l) = \frac{1}{\sum_{k=0}^{n} a_k p_k(l)} \quad \textit{for } l = 0, 1, \ldots, n. \tag{2}$$

Proof. By noting the relations $D_i = \sum_{j=0}^{n} p_i(j) E_j$ and $E_i = (1/v) \sum_{j=0}^{n} q_i(j) D_j$, we have

$$\sum_{i=0}^{n} b_i D_i = \left(\sum_{k=0}^{n} a_k D_k \right)^{-1} = \left(\sum_{k=0}^{n} \sum_{j=0}^{n} a_k p_k(j) E_j \right)^{-1}$$

$$= \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_k p_k(j) \right)^{-1} E_j = \frac{1}{v} \sum_{i=0}^{n} \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_k p_k(j) \right)^{-1} q_j(i) D_i.$$

Thus we obtain (1). Furthermore, by using $\sum_{i=0}^{n} q_j(i) p_i(l) = v \delta_{jl}$, we have

$$\sum_{i=0}^{n} b_i p_i(l) = \frac{1}{v} \sum_{i=0}^{n} \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_k p_k(j) \right)^{-1} q_j(i) p_i(l)$$

$$= \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_k p_k(j) \right)^{-1} \cdot \delta_{jl} = \left(\sum_{k=0}^{n} a_k p_k(l) \right)^{-1}.$$

Thus, the proposition is proved. □

In the case of a Hamming scheme, $p_k(j)$ and $q_k(j)$ are *Krawtchouk polynomials* defined by

$$p_k(j) = q_k(j) = P_k(j; n) = \sum_{i=0}^{\min(j,k)} (-q)^i (q-1)^{k-i} \binom{j}{i} \binom{n-i}{k-i}. \tag{3}$$

By using (1) and (3), we can easily obtain the simple explicit formula of $(\sum_{k=0}^{n} x^k D_k)^{-1}$.

Theorem 1. *For the Hamming scheme* $\langle D_0, \ldots, D_n \rangle$ *on* F^n, *let* $(\sum_{k=0}^{n} x^k D_k)^{-1} = \sum_{i=0}^{n} b_i D_i$; *then*

$$b_i = \frac{(-x)^i \{1 + x(q-2)\}^{n-i}}{(1-x)^n \{1 + x(q-1)\}^n} \tag{4}$$

holds for $i = 0, 1, \ldots, n$.

Proof. By (3), we have

$$\sum_{k=0}^{n} x^k p_k(j) = \sum_{k=0}^{n} x^k \sum_{i=0}^{j} (-q)^i (q-1)^{k-i} \binom{j}{i} \binom{n-i}{k-i}$$

$$= \sum_{i=0}^{j} \sum_{k'=0}^{n-i} (-q)^i (q-1)^{k'} \binom{j}{i} \binom{n-i}{k'} x^{k'+i}$$

$$= \{1 + x(q-1)\}^{n-j} (1-x)^j.$$

Then by using (1) and (3),

$$b_i = \frac{1}{q^n}\sum_{j=0}^{n}\frac{1}{(1-x)^j\{1+x(q-1)\}^{n-j}}\sum_{l=0}^{i}(-q)^l(q-1)^{j-l}\binom{i}{l}\binom{n-l}{j-l}$$

$$= \frac{1}{q^n}\frac{1}{\{1+x(q-1)\}^n}\sum_{l=0}^{i}\sum_{j'=0}^{n-l}(-q)^l(q-1)^{j'}\binom{i}{l}\binom{n-l}{j'}\left\{\frac{1+x(q-1)}{1-x}\right\}^{j'+l}$$

holds. After a straightforward but somewhat tedious calculation, we obtain (4). □

Now we shall consider the inverse matrix of $D_0 + xD_1$. In the case of $\sum_{i=0}^{n} b_i D_i = (D_0 + xD_1)^{-1}$, the explicit formula of b_i is very complicated, thus it is not easy to evaluate the value of b_i. In the following theorem, we claim that certain linear combinations for b_i are represented by simple formulas. In some cases, those formulas are useful in the statistical design of experiments (see [4]).

Theorem 2. *For the Hamming scheme $\langle D_0, \dots, D_n\rangle$ on F^n, let $(D_0 + xD_1)^{-1} = \sum_{i=0}^{n} b_i D_i$, then*

$$\sum_{i=m}^{n} b_i\binom{n-m}{i-m}(q-1)^{i-m} = \frac{m!(-x)^m}{\prod_{k=0}^{m}\{1+nx(q-1)-kxq\}}$$

holds for $m = 0, 1, \dots, n$.

Proof. Let $g(m) = \sum_{i=m}^{n} b_i\binom{n-m}{i-m}(q-1)^{i-m}$. When $m = 0$, let $l = 0$, $a_0 = 1$, $a_1 = x$ and $a_2 = \cdots = a_n = 0$ in (2), then it is easy to show that

$$g(0) = \sum_{i=0}^{n} b_i\binom{n}{i}(q-1)^i = \frac{1}{1+nx(q-1)}, \tag{5}$$

since $p_i(0) = \binom{n}{i}(q-1)^i$ holds.

Now, let $\langle D_0^{(m)}, \dots, D_{n-m}^{(m)}\rangle$ be a Hamming scheme on F^{n-m} and let

$$Z^{(m)} = (D_0^{(0)} + xD_1^{(0)})^{-1}\cdot\left\{\left(\bigotimes_{i=1}^{m} \boldsymbol{e}\right)\otimes \mathbf{1}_{q^{n-m}}\right\} = \left(\sum_{i=0}^{n} b_i D_i^{(0)}\right)\cdot\left\{\left(\bigotimes_{i=1}^{m} \boldsymbol{e}\right)\otimes \mathbf{1}_{q^{n-m}}\right\},$$

where $\boldsymbol{e} = (0, \dots, 0, 1)'$ is a q-dimensional vector, $\mathbf{1}_{q^{n-m}}$ is the q^{n-m}-dimensional all-one column vector, $\otimes$ indicates the direct product and $\otimes_{i=1}^{m}\boldsymbol{e} = \underbrace{\boldsymbol{e}\otimes\cdots\otimes\boldsymbol{e}}_{m}$.

For any two matrices A and B, let $\boldsymbol{S}$ be the set of the all ordered $(l+h)$-tuples consisting of l A's and hB's and define the following function:

$$f(A, B; l, h) = \sum_{(S_1, \dots, S_{l+h})\in \boldsymbol{S}} \bigotimes_{i=1}^{l+h} S_i.$$

Since $D_i^{(j-1)} = (J_q - I_q) \otimes D_{i-1}^{(j)} + I_q \otimes D_i^{(j)}$ holds for $0 \leqslant i \leqslant n-j+1$, where $D_{-1}^{(j)} = D_{n-j+1}^{(j)} = 0$, we obtain

$$\sum_{i=0}^{n} n_i D_i^{(0)} = \sum_{i=1}^{n} b_i (J_q - I_q) \otimes D_{i-1}^{(1)} + \sum_{i=0}^{n-1} b_i I_q \otimes D_i^{(1)} = \cdots$$

$$= \sum_{k=0}^{m} \left\{ f(J_q - I_q, I_q; m-k, k) \otimes \sum_{i=m-k}^{n-k} b_i D_{i-m+k}^{(m)} \right\},$$

where I_q is the $q \times q$ identity matrix and J_q is the $q \times q$ all-one matrix.

Furthermore, since $(A_1 \otimes \cdots \otimes A_m) \cdot (B_1 \otimes \cdots \otimes B_m) = A_1 B_1 \otimes \cdots \otimes A_m B_m$ holds for any matrices $\{A_i\}$ and $\{B_i\}$,

$$f(J_q - I_q, I_q; m-k, k) \left(\bigotimes_{i=0}^{m} \boldsymbol{e} \right) = f(\boldsymbol{j} - \boldsymbol{e}, \boldsymbol{e}; m-k, k)$$

holds, where $\boldsymbol{j}$ is the q-dimensional all-one column vector. Thus, by noting that $D_i^{(m)} \mathbf{1}_{q^{n-m}} = \binom{n-m}{i} (q-1)^i \mathbf{1}_{q^{n-m}}$ holds, we can rewrite $Z^{(m)}$ as follows:

$$Z^{(m)} = \sum_{k=0}^{m} \left\{ f(\boldsymbol{j} - \boldsymbol{e}, \boldsymbol{e}; m-k, k) \otimes \sum_{i=m-k}^{n-k} b_i \binom{n-m}{i-m+k} (q-1)^{i-m+k} \mathbf{1}_{q^{n-m}} \right\}.$$

On the other hand, let $\bar{\boldsymbol{e}} = (1, 0, \ldots, 0)'$, then similarly we obtain

$$0 = \left\{ \left(\bigotimes_{i=1}^{m} \bar{\boldsymbol{e}}' \right) \otimes \mathbf{1}'_{q^{n-m}} \right\} \cdot \left\{ \left(\bigotimes_{i=1}^{m} \boldsymbol{e} \right) \otimes \mathbf{1}_{q^{n-m}} \right\}$$

$$= \left\{ \left(\bigotimes_{i=1}^{m} \bar{\boldsymbol{e}}' \right) \otimes \mathbf{1}'_{q^{n-m}} \right\} \cdot (D_0^{(0)} + x D_1^{(0)}) Z^{(m)}$$

$$= \left\{ \left(\bigotimes_{i=1}^{m} \bar{\boldsymbol{e}}' \right) \otimes \mathbf{1}'_{q^{n-m}} \right\}$$

$$\times \left\{ x f(J_q - I_q, I_q; 1, m-1) \otimes D_0^{(m)} + \left(\bigotimes_{i=1}^{m} I_q \right) \otimes (D_0^{(m)} + x D_1^{(m)}) \right\} Z^{(m)}$$

$$= x (f(\boldsymbol{j}' - \bar{\boldsymbol{e}}', \bar{\boldsymbol{e}}'; 1, m-1) \otimes \mathbf{1}'_{q^{n-m}}) Z^{(m)}$$

$$+ \{1 + x(n-m)(q-1)\} \left\{ \left(\bigotimes_{i=1}^{m} \bar{\boldsymbol{e}}' \right) \otimes \mathbf{1}'_{q^{n-m}} \right\} Z^{(m)}. \tag{6}$$

It is obvious that

$$f(\boldsymbol{j}' - \bar{\boldsymbol{e}}', \bar{\boldsymbol{e}}'; 1, m-1) \cdot f(\boldsymbol{j} - \boldsymbol{e}, \boldsymbol{e}; m-k, k) = \begin{cases} m(q-2) & \text{if } k = 0, \\ m & \text{if } k = 1, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

holds. Thus, by using (7) and

$$\binom{n-m+1}{i-m+1} = \binom{n-m}{i-m+1} + \binom{n-m}{i-m},$$

the first term of the right-hand side of (6) is

$$(f(j' - \bar{e}', \bar{e}'; 1, m-1) \otimes \mathbf{1}'_{q^{n-m}}) Z^{(m)} = q^{n-m}\{-m \cdot g(m) + m \cdot g(m-1)\},$$

and similarly the second term is

$$\left\{\left(\bigotimes_{i=1}^{m} \bar{e}'\right) \otimes \mathbf{1}'_{q^{n-m}}\right\} Z^{(m)} = q^{n-m} \cdot g(m).$$

Therefore by (6), we have

$$\{1 + nx(q-1) - mxq\} \cdot g(m) = -mx \cdot g(m-1),$$

which proves the theorem together with (5). □

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References

[1] E. Bannai and T. Ito, Algebraic Combinatrics I (Benjamin, Menlo Park, CA, 1984).
[2] R.C. Bose and D.M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced designs, Ann. Math. Statist. 30 (1959) 21–38.
[3] F.J. MacWilliams and N.J.A. Sloane, The Theory of Error-Correcting Codes (North-Holland, New York, 1977).
[4] M. Mishima and M. Jimbo, Optimum orthogonal designs with multilevel-factors for correlated errors, submitted.
[5] J. Ogawa, The theory of the association algebra and the relationship algebra of a partially balanced incomplete block design, Institute of Stat. Mimeo. Ser. No. 224, Univ. of North Carolina, Chapel Hill, NC, April 1959.
[6] S. Yamamoto, Y. Fujii and N. Hamada, Composition of some series of association algebras, J. Sci. Hiroshima Univ. Ser. A-I 29 (1965) 181–215.